Recreational Mathematics with the len of operational research
The monkey and the coconuts ingenuity test

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Abstract: "The monkey and the coconuts" is a well-known problem of *Diophantine analysis*, staple in *recreational mathematics* collections and notorious for its difficulty. In this work, we feature the original and the Williams' versions and indicative solutions. Subsequently the problem is generalized and the implemented, programmed solving algorithms are presented. The numbers of *seafarers*, *remaining coconuts* left *over the morning after dividing into equal shares (zero or more),* and coconuts *giving to monkey in each division are control variables.* In this respect, old technique *diofantine equation is combined with new one, the linear integer programing, artificial intelligent and heuristic* algorithms. Our intention is to evaluate ingenuity, puzzles' solvings skill and to promote solvers to problem generators utilizing programming and operational research tools

"For the love of Mike, how many coconuts?
Hell poping around here",
Horace Lorimer
Editor-in-chief of the Saturday Evening Post

Background [1, 7]

The Williams version

The first description of the problem appeared in *Lewis Carroll*'s diaries in 1888: it involves a pile of nuts on a table serially divided by four brothers, each time with remainder of one given to a monkey, and the final division coming out even. A similar problem presented in *W.W. Rouse Ball's Elementary Algebra* (1890). The problem became famous when American novelist *Ben Ames Williams* modified an older problem and included it in a story entitled «Coconuts", in the October 9, 1926, issue of the *Saturday Evening Post.* Here is how Williams (condensed and paraphrased) stated the problem:

> *Five men and a monkey were shipwrecked on an island. They spent the first day gathering coconuts for food.During the night, one man woke up, and decided to take his share early. Therefore, he divided the coconuts in five piles. He had one coconut left over, and he gave that to the monkey. Then he hid his pile, put the rest back together.By, by each of the five men woke up, and did the same thing, one after the other: each one taking a fifth of the coconuts that were in the pile when he woke up, and having one left over for the monkey. In the morning, they divided what coconuts were left, and they came out in five equal shares. Of course, each one must have known there were coconuts missing; but each one was guilty as the others, so they did not say anything.*
> *How many coconuts were there in the original pile?*
>
> *Saturday Evening post 1926 Oct 9: Ben Ames Williams*

Williams had not included an answer in the story. More than 2,000 letters pleading for an answer to the problem inundated the magazine

Table 1 variables notation description and default values

symbol	default	description
n		*The number of coconuts*
m	*5*	*The number of sailors*
k	*1*	*The number of coconuts given to monkey in each division*
c or coc	*1*	*The number of coconuts left over the morning after divining*
f		*the number of coconuts of each sailor shared in the morning*
r	*m-1*	
rr	2(m mod 2)-1	*odd m rr=1: even m rr=-1*

The original version

Martin Gardner featured the problem in his April 1958 *Mathematical Games column in Scientific American* [3]. According to Gardner, Williams had modified an older problem to make it more confounding. In the older version, there is a coconut for the monkey on the final division; in Williams's version, the final division in the morning comes out even. The original story containing the problem was reprinted in full in Clifton Fadiman's [4] 1962 anthology *The Mathematical Magpie*, a book that the *Mathematical Association of America* recommends for acquisition by undergraduate mathematics libraries.

Solution

The most genious solution is based on the rule of division: *D=dp+r.* We apply this for m=5 (#seafarers) succesive divisions and one more for the final division on morning. Adding 4 to both sides of initial equation, we conclude that (#conconuts +4) is expressed as $\mathbf{m^{m+1}}$ while in Williams version is $\mathbf{m^{m}}$

$$n = 5p_i + 1 : n + 4 = 5(p_i + 1) : n + 4 = 5\frac{5p_2 + 1}{4} + 1 \rightarrow n + 4 = \frac{5(5p_2 + 1 + 4)}{4}$$

$$Old\ version\ n + 4 = \frac{5^2(p_2 + 1)}{4} \rightarrow n + 4 = 5^2 k_2 \rightarrow n + 4 = 5^6 \rightarrow n = 5^6 - 4 \rightarrow n = 15621$$

$$Williams\ version: n + 4 = \frac{5^4(5p_4 + 0)}{4} = 5^5 \ \ (\rightarrow p4 \propto 4) \rightarrow n + 4 = 3125 \ \ \rightarrow n = 3121$$

The aformetioned logical process could be fomalized as follows:

$$n = m^{m+g} - (m - 1) - km^m : gardner\ g = 1 : williams\ g = 0 \ \ (m = even\ and\ g = 0, k = 1)$$

The following table summarizes examples of the alternate variations

Table 2 function solving alternative old senarios

#Seafarers: m	(odd/even)	morning g	k	function	Value
5	odd	1	0	m^{m+g}-(m-1) –km^m	15621
5	odd	0			3121
4	even	1			1020
4	even	0	1		756

On problem generation

There are several problem and solution variations incorporating instead of constants some user defined variables as for example #seafarers and/or the #coconuts giving to the monkey [6]. In our work we incorpoate *all* variables to be user defined and we develop the corresponding solver

The problem can be generalized as follows:The numbers of *seafarers*, *remaining coconuts left over the morning after dividing into equal shares (zero or more), coconuts giving to monkey in each division could be user defined in other words, control variables.*

The proposed workflow includes:
The formulation of recursive formula of the #coconuts to be shared in each division,
the formulation of the linear integer diophantic equation (diophantine) with variables the n: initial #coconuts and f: the #coconuts of each sailor shared in the morning,
The solution utilizing extended euklidean algorithm, heuristics and integer programming and the programmed solver.

Recursive formula of the #coconuts n_i to be shared in each division,

First sailor leaves $n_1 = \frac{r^1(n-k)}{m^1} = \frac{r^1 n - r^1 k x_1}{m}$ to the others

so n_ι sailor leaves $n_i = \frac{r^i n - x_i}{m^i} \; : x_i = r(x_{i-1} + k m^{i-1}) \quad eq1$

i.e for m=5, c=1 k=1 and r=4 we have x_i = {4; 36; 244; 1476; 8404}

& fifth sailor leaves: $n_5 = \frac{4^5 n - 8404}{5^5} = \frac{1024n - 8404}{3125}$ to all

*Code x (0) = 0 : For i = 1 To m: x(i) = r * (x(i - 1) + k * ex(m, i - 1)) : next i*

Formalizing the diophantine equation **[2]**

According to the eq1 formula in the morning the #coconuts were leaved from 5th sailor and shared by 5 sailors are $5f = \frac{1024n - 8404}{3125}$. In general, case if c are the coconuts left over the morning after diving the diofantine equation will be s formulated as follows:

$$\frac{1024n - 8404}{3125} = 5f + c \Rightarrow 1024n - 8404 = 15625f + 3125c \rightarrow 1024n - 15625f - 8404 - 3125c = 0$$

The problem should be generated with a diophantine equation in the form of $an - bf - e = 0$ where

a is the coefficient of *n* the initial #coconuts : **a = *(m-1)^m***

b is the coefficient of f the seafarer share after diving in the morning ***b=m^m***

and the constant ***e= x (m, k) + m^m c***

In the followng table 3 diferent senarios figures confirm that constant e depends on k and c variables while a and b coefficients depend only on m.

m	k	c	e1 x(m,k)	a $(m-1)^m$	b m^{m+1}	e2 m^m c	e e1+e2	*equation*	n	f
5	1	1	8404	1024	15625	3125	11529	1024n- 15625f - 11529= 0	15621	1023
5	3	1	25212				28347	1024n- 15625f - 28337= 0	6238	407
5	4	3	33616			9375	42991	1024n- 15625f - 42991= 0	3109	201
6	2	4	310310	15625	279936	186624	496934	15625n- 279936f - 496934= 0	93302	5206

` **Table 3: Diofantine equation for diferent values of control variables**

Solving the equation: an-bf-c=0

We are going now to solve the above equation or the equivalent ***an-bf=gcd (a, b) i.e an-bf=1***

As example we use the case m=5, k=1 and c=1 => $1024n - 15625f$=1

To solve the equation the extended euclidean algorithm computing gcd (great common divisor) is used. The continued fraction expansion for $\frac{b}{a} = \frac{15625}{1024}$ is [15, 3, 1, 6, 2, 1, 3, 2, 1] and s it can be seen during the execution of algorithm related function loop.

Hence the convergents to given b/a is:

$$convergents\left(\frac{15625}{1024}\right) = [\frac{15}{1}\ \frac{46}{3}\ \frac{61}{4}\ \frac{412}{27}\ \frac{885}{58}\ \frac{1297}{85}\ \frac{4776}{313}\ \frac{k_f{:}10849}{k_n{:}711}].$$

we have $ak_f - bk_n = 1 \rightarrow 1024\text{x}10849 - 15625\text{x}711 = 1. \rightarrow \text{e}(ak_f - bk_n) - e = 0$

We may use the nominator and denominator of the last convegent quotient (k_f, k_n) to calculate n, f corespondigly

$$n = ek_f \, mod \, b \ \rightarrow n = 11529x10849 \, mod \, 15625 \ \rightarrow n = 15621$$

$$f = ek_n \, mod \, a \rightarrow f = 11529x711 \, mod \, 1024 \rightarrow f = 1023$$

Programming the algorithm

The algorithm is programmed to find n #initial coconuts for given values *of m, k and c.* A program screenshot is given in figure-1. Furthermore, the prblem is formulated as a linear integer-programming model to minimize. Finally another program using heuristics mimics the human algorithmic process is developed.

	#sailors m	On each division leaves k	Left on morning c							1	1	
	7	2	3									
clear	Diofantine equation	generate constant value : x(m,k)+c*m^m								coefn	coeff	coefc
									2470629			
CommandBu	279936n- 5764801f - 8993913= 0	12	156	1524	13260	108372	851916	6523284	8993913	279936	5764801	8993913
	4941246	239943										
			4941246	4941244	705892	2	1					
			4235352	4235350	605050	2	1					
4941258			3630300	3630298	518614	2	1					
			3111684	3111682	444526	2	1					
			2667156	2667154	381022	2	1					
			2286132	2286130	326590	2	1					
			1959540	1959538	279934	2	1					
			1679604	1679601	239943	3	7					
				239943	4941246							
	continious fraction expansion	20	1	1	2	5	1	1	4			

figure 1: progam screenshot

Conclusions

The particular famous puzzle remains an excellent test of advanced math and logic skills, and is for years a chalenge for parameterization, algorithmization and programming. The presented problem formulation is really opened and easily can be transformed for additional generalization, as for example different #coconuts given to monkey in each division or finding of #seafarers for given #coconuts. The potential of linear integer programming to solve the problem minimizing the #coconuts, subject of problem constraints is demonstrated. Finaly the agnostic problem solution using heuristics was a programming chalenge.

References

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7.Claude Mydorge (1585–1647) was born in Paris and trained as a lawyer. But as a young man, he also studied geometry and optics and his interests turned to mathematics, particularly geometry. Mathematical recreations in the form of problem solving were popular in France during Mydorge's time and he became involved in the movement. His collection of proposed problems, *Examen du livre des récréations mathématiques ...* (1639), had a large appeal and became the model for similar book